

Topic Study Group 13

Teaching and Learning of Calculus

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1. Aims of the TSG

This Topic Study Group sought contributions on research and development in the teaching and learning of Calculus, both at the upper secondary and tertiary levels. Contributions accounted for advances, new trends, and important work done in recent years on the teaching and learning processes of Calculus. These included:

- Introducing and building basic concepts of Calculus in upper secondary education,
- Meeting the challenges of teaching and learning Calculus and Analysis at universities and through online courses,
- Teaching and learning of Calculus for special audiences (*e.g.* professional training, engineering, life sciences),
- Use of technology in the teaching and learning of Calculus, including online courses
- The role of visualisation in the teaching and learning of Calculus,
- Analysis of textbooks concerning the presentation of the concepts of Calculus and Analysis,
- Easing the transition between secondary and tertiary education in the teaching and learning of Calculus, and between Calculus and Analysis at the tertiary level,
- Theoretical approaches to study the phenomena related to the teaching and learning of Calculus.

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Contributions also described theoretical or pragmatic research into effective practices for the teaching and learning of key concepts of Calculus such as co-variation of functions, limits, continuity, differentiation, integration, or the Fundamental Theorem of Calculus, among others.

1.1. Submissions

We received 36 submissions from 19 countries (South America: 6; North America: 10; East Asia: 7; Southeast Asia: 1; South Asia: 2, Europe: 5; Middle East and North Africa: 5), thus reaching our goal of diverse cultural representation. Of those 36 submissions, eleven were accepted for long paper presentations, nineteen for short paper presentations, and eight as posters. None were rejected. One presenter selected for a long paper and one presenter selected for a short paper withdrew after the congress was rescheduled for 2021.

1.2. Sessions

Because almost all presenters were joining remotely and representing many different time zones, the presentations were grouped geographically, first those from East and South Asia, then those from Europe, the Middle East, and North Africa, and finally those from the Americas. Each session began with three or four 15-minute presentations, followed by five or six 5-minute presentations. There was little time for discussion.

1.3. Paper Topics

A list of the papers and authors are included in order of presentation and are organized in Tab. 1 (on the next page).

2. Conference Themes

There were three main themes for the papers presented in this topic study group. The first dealt with student understandings and misunderstandings of basic concepts of calculus. These included rate of change, limits, continuity, derivatives, differentials, and definite integrals. At a more basic level, there was also discussion of how to improve the covariational reasoning of students and general student difficulties with the language of mathematics and how mathematics uses language.

A second theme turned to the use of technological tools to help students build understanding of certain fundamental ideas.

Quite a few of the presentations focused on the third theme, presenting a variety of techniques for improving instruction in the calculus classroom. Several described the use of inquiry-based learning. There was discussion of other approaches to creating an active learning environment as well as the use of flipped instruction, an emphasis on modeling, and the use of writing assignments.

Other topics included a comparison of textbooks, a comparison of how physics and mathematics differ in their approach to ordinary differential equations, the

introduction of tangents and asymptotes without reference to limits, and a discussion of how to deal with student overgeneralization of the concept of linearity.

Tab. 1. List of papers presented

Paper and author(s)
[1] Mathematical knowledge for teaching of calculus: an exploratory study of secondary school teachers mathematical thinking related to concepts in calculus. Jonaki B Ghosh (India).
[2] Modeling concepts of derivative and differential with educational software. Vladimir Nodelman (Israel).
[3] Constructing knowledge using digital tools: the case of the inflection point. Regina Ovoenko and Anatoli Kouropatov (Israel).
[4] Students' interpretations of the definite integral. Inen Akrouti (Tunisia).
[5] Comparison of mathematics textbooks in IB school and Chinese public high school: take core concept — calculus as an example. Yun Lu (China).
[6] Research in calculating areas between curves. Gordana Stankov and Djurdjica Takaci (Serbia).
[7] The concept of continuity through different types of representations of the function. Matthias Antonopoulos and Leonora Antonopoulou (Greece).
[8] Actions in the learning environment: analyzing physics and mathematics lessons in the case of ODE. Kristina Elisabeth Juter, Örjan Hansson, and Andreas Redfors (Sweden).
[9] From upper secondary school to university calculus: language difficulties versus conceptual difficulties. Arne Hole, Inger Christin Borge, and Liv Sissel Grønmo (Norway).
[10] The discrete-dense-continuous phenomenon and its implication in continuous. Elizabeth Montoya Delgadillo (Chile).
[11] A limit free calculus for introducing the concepts of tangent and asymptote. an educational proposal inspired by the past. Maria Astrid Cuida Gomez (Spain).
[12] An approach to reduce the number of failure students in a large calculus class. Jianhui Pan (China).
[13] The exponential function from the viewpoint of mathematical modelling: a Chilean lesson study. Carlos Andres Ledezma Araya and Elizabeth Montoya Delgadillo (Chile).
[14] Using open education resources to promote the active learning of calculus in urban districts. Kenneth Horwitz (USA).
[15] Mathematics anxiety levels among students in an inquiry-based calculus class. Harman Prasad Aryal and Otto Joshua Shaw (Nepal).
[16] Learning difficulties in calculus: an investigation through students' written solutions. Raquel Carneiro Dorr (Brazil).
[17] The design and use of low instructional overhead tasks in undergraduate calculus: making student reasoning more accessible to calculus instructors. David C. Webb (USA).
[18] The observed impact implementing inquiry-based learning at a calculus classroom. Su Liang (China).
[19] Teaching calculus based on complexity theory of teaching and learning. Mehmet Turegun (USA).
[20] Notions of continuity of the pre-service teachers: reflections for a problematization. Antonio Bonilla and Ricardo Cantoral (Mexico).
[21] Resignification of the derivative in a school situation with a perspective of an exclusion-inclusion dialectic: from emulation of the concept to autonomy of uses. Jose Luis Morales Reyes and Francisco Cordero Osorio (Mexico).
[22] Covariational reasoning: an axis in the construction process of the definite integral concept. Mihaly Andre Martinez Miraval and Martha Leticia Garcia Rodriguez (Peru).
[23] The “overgeneralization of linearity”: difficulty, conflict or obstacle? Nicolas Lopez and Gloria Ines Neira Sanabria (Colombia).
[24] Rate of change: meanings students have in accordance with context. Dafna Elias, Tommy Dreyfus, Anatoli Kouropatov, and Noah Sella (Israel).

3. Areas for Future Research

None of the presentations specifically addressed the problems of preparation for calculus. This is a huge issue, especially in places such as the United States where student preparation for university is so varied in quality and so highly correlated with socio-economic status. Good work is being done in trying to address these disparities. They require attention from the research community to understand what works in what situations and why.

Student understanding and misunderstanding of the concepts of calculus has been a rich source for research in the teaching and learning of calculus. Thirty years ago, the focus was on how students misconceive so many of these fundamental ideas. Within the past decade, this has shifted to a more productive line of exploring natural student understandings that can be encouraged and developed to improve student grasp of and ability to use fundamental aspects of calculus. Good examples of this are the development of covariational reasoning and the development of an understanding of limits expressed in terms of narrowing bounds on the distance from the target value, placing the emphasis on what happens to the dependent variable rather than the independent variable. There is still work to be done in identifying productive approaches to basic ideas of calculus and understanding how they can be effectively encouraged.

Technology in its many forms is a constant presence. While most of the work has focused on exhibiting the effectiveness of a clever new tool, much more work needs to be done on how to balance the use of what has become basic and ubiquitous technology such as computer algebra systems. How have they changed what students need to learn and be able to carry with them beyond the calculus class? What procedures for differentiation or integration are still essential and why?

Finally, as the emphasis on improved approaches to teaching and learning in this topic study group has revealed, there is a great deal of work being done on the implementation of a variety of active learning approaches. There is no question that when undertaken by a dedicated and enthusiastic instructor, these can greatly improve student outcomes. The questions that require exploration revolve around how these efforts can be scaled up. How does one convince reluctant colleagues to attempt active approaches to their teaching? What kinds of supports are most helpful? How can departments deal with the fact that later adopters are often discouraged by the difficulties they encounter? What are the ingredients of active learning that are easiest to implement on a broad scale and most effective? These are very broad questions with applications to any aspect of the teaching and learning of mathematics, but there is good research into a variety of approaches to active learning that apply specifically to the context of calculus instruction. Narrowing the focus in this way promises to generate good ideas and significantly improve the teaching and learning of calculus.